

# Parabolic integrodifferential identification problems related to radial memory kernels II\*

Alberto Favaron (Milan), Alfredo Lorenzi (Milan) <sup>†</sup>

**Abstract.** We are concerned with the problem of recovering the radial kernel  $k$ , depending also on time, in the parabolic integro-differential equation

$$D_t u(t, x) = \mathcal{A}u(t, x) + \int_0^t k(t-s, |x|) \mathcal{B}u(s, x) ds + \int_0^t D_{|x|} k(t-s, |x|) \mathcal{C}u(s, x) ds + f(t, x),$$

$\mathcal{A}$  being a uniformly elliptic second-order linear operator in divergence form. We single out a special class of operators  $\mathcal{A}$  and two pieces of suitable additional information for which the problem of identifying  $k$  can be uniquely solved locally in time when the domain under consideration is a ball or a disk.

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## 1 Posing the identification problem

The present paper is strictly related to our previous one [3]. Indeed, the problem we are going to investigate consists, as in [3], in identifying an unknown radial memory kernel  $k$  also depending on time, which appears in the following integro-differential equation related to the ball  $\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x| < R\}$ ,  $R > 0$  and  $|x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ :

$$D_t u(t, x) = \mathcal{A}u(t, x) + \int_0^t k(t-s, |x|) \mathcal{B}u(s, x) ds + \int_0^t D_{|x|} k(t-s, |x|) \mathcal{C}u(s, x) ds + f(t, x),$$

$$\forall (t, x) \in [0, T] \times \Omega. \quad (1.1)$$

We emphasize that the aim of the present paper is to study the identification problem related to (1.1) when the domain  $\Omega$  is a *full* ball. This is exactly a singular domain for our problem as we noted in Remark 2.9 in [3], where we were able to recover the

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<sup>†</sup>The authors are members of G.N.A.M.P.A. of the Italian Istituto Nazionale di Alta Matematica (I.N.d.A.M.)

kernel  $k$  only in the case of a spherical corona or an annulus  $\Omega$ . In this paper we show that our identification problem can actually be solved in suitable weighted spaces if we appropriately restrict the class of admissible differential operators  $\mathcal{A}$  to a class whose coefficients have an appropriate structure in a neighbourhood of the centre  $x = 0$  of  $\Omega$ , which turns out to be a “singular point” for our problem.

In equation (1.1)  $\mathcal{A}$  and  $\mathcal{B}$  are two second-order linear differential operators, while  $\mathcal{C}$  is a first-order differential operator having the following forms, respectively:

$$\mathcal{A} = \sum_{j=1}^3 D_{x_j} \left( \sum_{k=1}^3 a_{j,k}(x) D_{x_k} \right), \quad \mathcal{B} = \sum_{j=1}^3 D_{x_j} \left( \sum_{k=1}^3 b_{j,k}(x) D_{x_k} \right), \quad \mathcal{C} = \sum_{j=1}^3 c_j(x) D_{x_j}. \quad (1.2)$$

In addition, operator  $\mathcal{A}$  has a very special structure, since its coefficients  $a_{i,j}$ ,  $i, j = 1, 2, 3$ , have the following particular representation, (cf. [3], formula (2.4), where  $(b, d)$  is changed in  $(-b, -d)$ ):

$$\begin{cases} a_{1,1}(x) = a(|x|) + \frac{(x_2^2 + x_3^2)[c(x) + b(|x|)]}{|x|^2} - \frac{x_1^2 d(|x|)}{|x|^2}, \\ a_{2,2}(x) = a(|x|) + \frac{(x_1^2 + x_3^2)[c(x) + b(|x|)]}{|x|^2} - \frac{x_2^2 d(|x|)}{|x|^2}, \\ a_{3,3}(x) = a(|x|) + \frac{(x_1^2 + x_2^2)[c(x) + b(|x|)]}{|x|^2} - \frac{x_3^2 d(|x|)}{|x|^2}, \\ a_{j,k}(x) = a_{k,j}(x) = -\frac{x_j x_k [b(|x|) + c(x) + d(|x|)]}{|x|^2}, \quad 1 \leq j, k \leq 3, \quad j \neq k, \end{cases} \quad (1.3)$$

where the functions  $a, b, c, d$  are *non-negative* and enjoy the following properties:

$$a, b, d \in C^2([0, R]), \quad c \in C^2(\overline{\Omega}), \quad (1.4)$$

$$a(r) > d(r), \quad \forall r \in [0, R], \quad b(0) + c(0) = 0, \quad d(0) = 0. \quad (1.5)$$

In particular, we note that each coefficient  $a_{i,j}$  is Lipschitz-continuous in  $\overline{\Omega}$ .

We now introduce the function  $h$  defined by

$$h(r) = a(r) - d(r), \quad \forall r \in [0, R], \quad (1.6)$$

and which is non-negative by virtue of (1.5). Then, as we noted in [3], for every  $x \in \overline{\Omega}$  and  $\xi \in \mathbb{R}^3$  we have

$$\begin{aligned} \sum_{j,k=1}^3 a_{j,k}(x) \xi_j \xi_k &\geq a(|x|) |\xi|^2 + \frac{b(|x|) + c(x)}{|x|^2} |x \wedge \xi|^2 - \frac{d(|x|)}{|x|^2} [x \cdot \xi]^2 \\ &\geq a(|x|) |\xi|^2 + \frac{b(|x|)}{|x|^2} |x \wedge \xi|^2 - \frac{d(|x|)}{|x|^2} [x \cdot \xi]^2 \geq h(|x|) |\xi|^2 \geq 0, \end{aligned} \quad (1.7)$$

where  $\wedge$  and  $\cdot$  denote, respectively, the wedge and inner products in  $\mathbb{R}^3$ .

From (1.7) it follows that the condition of uniform ellipticity of  $\mathcal{A}$ , i.e.

$$\alpha_1 |\xi|^2 \leq \sum_{j,k=1}^3 a_{j,k}(x) \xi_j \xi_k \leq \alpha_2 |\xi|^2, \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^3, \quad (1.8)$$

is trivially satisfied with  $\alpha_1 = \min_{r \in [0, R]} h(r)$  and  $\alpha_2 = \|h + b\|_{C([0, R])} + \|c\|_{C(\overline{\Omega})}$ . Then we prescribe the *initial condition*:

$$u(0, x) = u_0(x), \quad \forall x \in \Omega, \quad (1.9)$$

$u_0 : \overline{\Omega} \rightarrow \mathbb{R}$  being a given smooth function, as well as one of the following boundary value conditions, where  $u_1 : [0, T] \times \overline{\Omega} \rightarrow \mathbb{R}$  is a given smooth function:

$$(D) \quad u(t, x) = u_1(t, x), \quad \forall (t, x) \in [0, T] \times \partial\Omega, \quad (1.10)$$

$$(N) \quad \frac{\partial u}{\partial n}(t, x) = \frac{\partial u_1}{\partial n}(t, x), \quad \forall (t, x) \in [0, T] \times \partial\Omega. \quad (1.11)$$

Here D and N stand, respectively, for the Dirichlet and Neumann boundary conditions, whereas  $n$  denotes the outwarding normal to  $\partial\Omega$ .

**Remark 1.1.** The conormal vector associated with the matrix  $\{a_{j,k}(x)\}_{j,k=1}^3$  defined by (1.3) and the boundary  $\partial\Omega$  coincides with  $R^{-1}[a(R) - d(R)]x$ , i.e. with the outwarding normal  $n(x)$ .

To determine the radial memory kernel  $k$  we need also the two following pieces of information:

$$\Phi[u(t, \cdot)](r) := g_1(t, r), \quad \forall (t, r) \in [0, T] \times (0, R), \quad (1.12)$$

$$\Psi[u(t, \cdot)] := g_2(t), \quad \forall t \in [0, T], \quad (1.13)$$

where, representing with  $(r, \varphi, \theta)$  the usual spherical co-ordinates with pole at  $x = 0$ ,  $\Phi$  and  $\Psi$  are two linear operators acting, respectively, on the angular variables  $\varphi, \theta$  only and all the space variables  $r, \varphi, \theta$ .

*Convention:* from now on we will denote by  $P(K)$ ,  $K \in \{D, N\}$ , the identification problem consisting of (1.1), (1.9), the boundary condition (K) and (1.12), (1.13).

An example of admissible linear operators  $\Phi$  and  $\Psi$  is the following:

$$\Phi[v](r) := \int_0^\pi \sin\theta d\theta \int_0^{2\pi} v(rx') d\varphi, \quad (1.14)$$

$$\Psi[v] := \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} \psi(rx') v(rx') d\varphi, \quad (1.15)$$

where  $x' = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$ , while  $\psi : \overline{\Omega} \rightarrow \mathbb{R}$  is a smooth assigned function.

**Remark 1.2.** We note that (1.14) coincides with (1.12) in [3] with  $\lambda = 1$ . We stress here that at present this case, along with the particular choice (1.3) of the coefficients  $a_{i,j}$ , seems to be the only one allowing an analytical treatment in the usual  $L^p$ -spaces when dealing with a full ball.

From (1.10) – (1.13) we (formally) deduce that our data must satisfy the following consistency conditions, respectively:

$$(C1,D) \quad u_0(x) = u_1(0, x), \quad \forall x \in \partial\Omega, \quad (1.16)$$

$$(C1,N) \quad \frac{\partial u_0}{\partial n}(x) = \frac{\partial u_1}{\partial n}(0, x), \quad \forall x \in \partial\Omega, \quad (1.17)$$

$$\Phi[u_0](r) = g_1(0, r), \quad \forall r \in (0, R), \quad (1.18)$$

$$\Psi[u_0] = g_2(0). \quad (1.19)$$

## 2 Main results

In this section we state our *local in time* existence and uniqueness result related to the identification problem P(K). For this purpose we assume that the coefficients of operator  $\mathcal{A}$  satisfies (1.3) – (1.5), whereas, as far as the coefficients  $b_{i,j}$  and  $c_i$  of operators  $\mathcal{B}$ ,  $\mathcal{C}$  are concerned, we assume:

$$b_{i,j} \in W^{1,\infty}(\Omega), \quad c_i \in L^\infty(\Omega), \quad i, j = 1, 2, 3. \quad (2.1)$$

In order to find out the right hypotheses on the linear operators  $\Phi$  and  $\Psi$ , it will be convenient to rewrite the operator  $\mathcal{A}$  in the spherical co-ordinates  $(r, \varphi, \theta)$ .

As a consequence, using representation (1.3) for the  $a_{i,j}$ 's, through lengthy but easy computations, we obtain the following polar representation  $\tilde{\mathcal{A}}$  for the second-order differential operator  $\mathcal{A}$ :

$$\begin{aligned} \tilde{\mathcal{A}} = & D_r [h(r) D_r] + \frac{2h(r) D_r}{r} + \frac{a(r) + b(r)}{r^2 \sin \theta} \left[ (\sin \theta)^{-1} D_\varphi^2 + D_\theta (\sin \theta D_\theta) \right] \\ & + \frac{1}{r^2 \sin \theta} \left[ (\sin \theta)^{-1} D_\varphi [\tilde{c}(r, \varphi, \theta) D_\varphi] + D_\theta (\tilde{c}(r, \varphi, \theta) \sin \theta D_\theta) \right], \end{aligned} \quad (2.2)$$

where we have set  $\tilde{c}(r, \varphi, \theta) = c(r \cos \varphi \sin \theta, r \sin \varphi \sin \theta, r \cos \theta)$ .

Before listing our requirements concerning operators  $\Phi$  and  $\Psi$  and the data, we recall (cf. [4]) some definitions about weighted Sobolev spaces. Given an  $n$ -dimensional domain  $\Omega$  the weighted Sobolev spaces  $W_\sigma^{k,p}(\Omega)$ ,  $k \in \mathbb{N}$ ,  $p \in [1, +\infty]$ ,  $\sigma \in \mathbb{R}$ , are defined by

$$W_\sigma^{k,p}(\Omega) = \left\{ f \in W_{loc}^{k,p}(\Omega \setminus \{0\}) : \|f\|_{W_\sigma^{k,p}(\Omega)} = \left( \sum_{0 \leq |\alpha| \leq k} \int_\Omega |x|^\sigma |D^\alpha f(x)|^p dx \right)^{1/p} < +\infty \right\}, \quad (2.3)$$

where

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \quad |\alpha| = \sum_{i=1}^n |\alpha_i|, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n}.$$

Of course,  $W_\sigma^{k,p}(\Omega)$  turns out to be a Banach space when endowed with the norm  $\|\cdot\|_{W_\sigma^{k,p}(\Omega)}$ . In particular, taking  $\sigma = 0$  in (2.3) we obtain the usual Sobolev spaces  $W^{k,p}(\Omega)$  whereas taking  $k = 0$  we obtain the weighted  $L^p$ -spaces defined by

$$L_\sigma^p(\Omega) = \left\{ f \in L_{loc}^p(\Omega) : \|f\|_{L_\sigma^p(\Omega)} = \left( \int_\Omega |x|^\sigma |f(x)|^p dx \right)^{1/p} < +\infty \right\}. \quad (2.4)$$

**Lemma 2.1.** *Operator  $\Phi$  defined by (1.14) maps  $W^{2,p}(\Omega)$  continuously into  $W^{2,p}_2(0, R)$ .*

*Proof.* Taking  $u \in W^{2,p}(\Omega)$  from (1.14) it follows that

$$D_r^{(j)}\Phi[u](r) = \Phi[D_r^{(j)}u](r), \quad \forall j = 0, 1, 2. \quad (2.5)$$

Hence, denoting with  $p'$  the conjugate exponent of  $p$ , from Hölder's inequality we obtain

$$\begin{aligned} \|\Phi[u]\|_{L^p_2(0,R)}^p &= \int_0^R r^2 |\Phi[u](r)|^p dr = \int_0^R r^2 \left| \int_0^\pi \sin\theta d\theta \int_0^{2\pi} u(rx') d\varphi \right|^p dr \\ &\leq (4\pi)^{p/p'} \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} |u(rx')|^p d\varphi = (4\pi)^{p/p'} \|u\|_{L^p(\Omega)}^p. \end{aligned} \quad (2.6)$$

Repeating similar computations and using the well-known inequalities

$$|D_r u(rx')| \leq |\nabla u(rx')|, \quad |D_r^2 u(rx')| \leq \sum_{j,k=1}^3 |D_{x_j} D_{x_k} u(rx')|^2, \quad (2.7)$$

from (2.5) we can easily find that the following inequalities hold:

$$\|D_r \Phi[u]\|_{L^p_2(0,R)}^p \leq C_1 \|u\|_{W^{1,p}(\Omega)}^p, \quad \|D_r^2 \Phi[u]\|_{L^p_2(0,R)}^p \leq C_2 \|u\|_{W^{2,p}(\Omega)}^p, \quad (2.8)$$

where  $C_1$  and  $C_2$  are two non-negative constants depending on  $p$  only.

Therefore, from (2.6) and (2.8) it follows that there exists a non-negative constant  $C_3$ , independent of  $u$ , such that

$$\|\Phi[u]\|_{W^{2,p}_2(0,R)} \leq C_3 \|u\|_{W^{2,p}(\Omega)}. \quad (2.9)$$

□

In this paper we will use Sobolev spaces  $W^{k,p}(\Omega)$  with

$$p \in (3, +\infty) \quad (2.10)$$

and we will assume that the functionals  $\Phi$  and  $\Psi$  satisfy the following requirements:

$$\Phi \in \mathcal{L}(L^p(\Omega); L^p_2(0, R)), \quad \Psi \in L^p(\Omega)^*, \quad (2.11)$$

$$\Phi[wu] = w \Phi[u], \quad \forall (w, u) \in L^p_2(0, R) \times L^p(\Omega), \quad (2.12)$$

$$D_r \Phi[u](r) = \Phi[D_r u](r), \quad \forall u \in W^{1,p}(\Omega) \text{ and } r \in (0, R), \quad (2.13)$$

$$\Phi \tilde{\mathcal{A}} = \tilde{\mathcal{A}}_1 \Phi \quad \text{on } W^{2,p}(\Omega), \quad (2.14)$$

$$\Psi \tilde{\mathcal{A}} = \Psi_1 \quad \text{on } W^{2,p}(\Omega), \quad \Psi_1 \in W^{1,p}(\Omega)^*, \quad (2.15)$$

where

$$\tilde{\mathcal{A}}_1 = D_r [h(r) D_r] + 2 \frac{h(r)}{r} D_r. \quad (2.16)$$

To state our result concerning the identification problem  $P(K)$ ,  $K \in \{D, N\}$ , we need to make also the following assumptions on the data  $f$ ,  $u_0$ ,  $u_1$ ,  $g_1$ ,  $g_2$ :

$$f \in C^{1+\beta}([0, T]; L^p(\Omega)), \quad f(0, \cdot) \in W^{2,p}(\Omega), \quad (2.17)$$

$$u_0 \in W^{4,p}(\Omega), \quad \mathcal{B}u_0 \in W_K^{2\delta,p}(\Omega), \quad (2.18)$$

$$u_1 \in C^{2+\beta}([0, T]; L^p(\Omega)) \cap C^{1+\beta}([0, T]; W^{2,p}(\Omega)), \quad (2.19)$$

$$\mathcal{A}u_0 + f(0, \cdot) - D_t u_1(0, \cdot) \in W_K^{2,p}(\Omega), \quad (2.20)$$

$$F := k'_0 \mathcal{C}u_0 + k_0 \mathcal{B}u_0 + \mathcal{A}^2 u_0 + \mathcal{A}f(0, \cdot) - D_t^2 u_1(0, \cdot) + D_t f(0, \cdot) \in W_K^{2\beta,p}(\Omega), \quad (2.21)$$

$$g_1 \in C^{2+\beta}([0, T]; L_2^p(0, R)) \cap C^{1+\beta}([0, T]; W_2^{2,p}(0, R)), \quad \frac{1}{r} D_t D_r g_1 C^\beta([0, T]; L_2^p(0, R)), \quad (2.22)$$

$$g_2 \in C^{2+\beta}([0, T]; \mathbb{R}), \quad (2.23)$$

where  $\beta \in (0, 1/2) \setminus \{1/(2p)\}$ ,  $\delta \in (\beta, 1/2) \setminus \{1/(2p)\}$  and function  $k_0$  in (2.21) is defined by formula (3.20). Moreover, the spaces  $W_K^{2,p}(\Omega)$  are defined by

$$W_K^{2,p}(\Omega) = \{w \in W^{2,p}(\Omega) : w \text{ satisfies the homogeneous condition (K)}\}, \quad (2.24)$$

whereas the spaces  $W_K^{2\gamma,p}(\Omega) \equiv (L^p(\Omega), W_K^{2,p}(\Omega))_{\gamma,p}$ ,  $\gamma \in (0, 1/2] \setminus \{1/(2p)\}$ , are interpolation spaces between  $W_K^{2,p}(\Omega)$  and  $L^p(\Omega)$  and they are defined [5, section 4.3.3], respectively, by:

$$W_D^{2\gamma,p}(\Omega) = \begin{cases} W^{2\gamma,p}(\Omega), & \text{if } 0 < \gamma < 1/(2p), \\ \{u \in W^{2\gamma,p}(\Omega) : u = 0 \text{ on } \partial\Omega\}, & \text{if } 1/(2p) < \gamma \leq 1/2, \end{cases} \quad (2.25)$$

$$W_N^{2\gamma,p}(\Omega) = W^{2\gamma,p}(\Omega), \quad \text{if } 0 < \gamma \leq 1/2. \quad (2.26)$$

**Remark 2.2.** Assumption (2.22) ensures that  $D_t \tilde{\mathcal{A}}_1 g_1 \in C^{2+\beta}([0, T], L_2^p(0, R))$  (see formula (3.16)).

**Remark 2.3.** Observe that our choice  $p \in (3, +\infty)$  implies the embeddings

$$W^{1,p}(\Omega) \hookrightarrow C^{(p-3)/p}(\overline{\Omega}), \quad (2.27)$$

$$W_2^{1,p}(0, R) \hookrightarrow C^{(p-3)/p}([0, R]). \quad (2.28)$$

In fact, while (2.27) is a classical consequence of the Sobolev embedding theorems ([1]), Theorem 5.4, (2.28) follows immediately from the inequalities

$$\begin{aligned} |u(t) - u(s)| &\leq \int_s^t \xi^{-2/p} \xi^{2/p} |u'(\xi)| d\xi \leq \left[ \int_s^t \xi^{-2/(p-1)} d\xi \right]^{1/p'} \|u'\|_{L_2^p(0, R)} \\ &\leq \left( \frac{p-1}{p-3} \right)^{1/p'} |t-s|^{(p-3)/p} \|u\|_{W_2^{1,p}(0, R)}, \quad \forall s, t \in [0, R] \end{aligned} \quad (2.29)$$

Assume also that  $u_0$  satisfies the following conditions for some positive constant  $m$ :

$$J_0(u_0)(r) := |\Phi[\mathcal{C}u_0](r)| \geq m, \quad \forall r \in (0, R), \quad (2.30)$$

$$J_1(u_0) := \Psi[J(u_0)] \neq 0, \quad (2.31)$$

where we have set:

$$J(u_0)(x) := \left( \mathcal{B}u_0(x) - \frac{\Phi[\mathcal{B}u_0](|x|)}{\Phi[\mathcal{C}u_0](|x|)} \mathcal{C}u_0(x) \right) \exp \left[ \int_{|x|}^R \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right], \quad \forall x \in \Omega. \quad (2.32)$$

**Remark 2.4.** According to (2.11) and (2.12) it follows that:

$$\Phi[J(u_0)](r) = \exp \left[ \int_r^R \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right] \Phi \left( \mathcal{B}u_0 - \frac{\Phi[\mathcal{B}u_0]}{\Phi[\mathcal{C}u_0]} \mathcal{C}u_0 \right)(r) = 0, \quad \forall r \in (0, R). \quad (2.33)$$

This means that operator  $\Psi$  *cannot be chosen of the form*  $\Psi = \Lambda\Phi$ , *where*  $\Lambda$  *is in*  $L_2^p(0, R)^*$ , i.e.  $\Lambda[v] = \int_0^R r^2 \rho(r) v(r) dr$  for any  $v \in L_2^p(0, R)$  and some  $\rho \in L_2^{p'}(0, R)$ , otherwise condition (2.31) would be not satisfied. In the explicit case, when  $\Phi$  and  $\Psi$  have the integral representation (1.14) and (1.15), this means that no function  $\psi$  of the form  $\psi(x) = |x|^2 \rho(|x|)$  is allowed.

**Remark 2.5.** When operators  $\Phi$  and  $\Psi$  are defined by (1.14), (1.15) conditions (2.30), (2.31) can be rewritten as:

$$\left| \int_0^\pi \sin \theta d\theta \int_0^{2\pi} \mathcal{C}u_0(rx') d\varphi \right| \geq m_1, \quad \forall r \in (0, R), \quad (2.34)$$

$$\begin{aligned} & \left| \int_0^R r^2 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} \psi(rx') \left( \mathcal{B}u_0(rx') - \frac{\int_0^\pi \sin \theta d\theta \int_0^{2\pi} \mathcal{B}u_0(rx') d\varphi}{\int_0^\pi \sin \theta d\theta \int_0^{2\pi} \mathcal{C}u_0(rx') d\varphi} \mathcal{C}u_0(rx') \right) \right. \\ & \quad \left. \times \exp \left[ \int_r^R \frac{\int_0^\pi \sin \theta d\theta \int_0^{2\pi} \mathcal{B}u_0(\xi x') d\varphi}{\int_0^\pi \sin \theta d\theta \int_0^{2\pi} \mathcal{C}u_0(\xi x') d\varphi} d\xi \right] d\varphi \right| \geq m_2 \end{aligned} \quad (2.35)$$

for some positive constants  $m_1$  and  $m_2$ .

Finally, we introduce the Banach spaces  $\mathcal{U}^{s,p}(T)$ ,  $\mathcal{U}_K^{s,p}(T)$  ( $K \in \{D, N\}$ ) which are defined for any  $s \in \mathbb{N} \setminus \{0\}$  by:

$$\begin{cases} \mathcal{U}^{s,p}(T) = C^s([0, T]; L^p(\Omega)) \cap C^{s-1}([0, T]; W^{2,p}(\Omega)), \\ \mathcal{U}_K^{s,p}(T) = C^s([0, T]; L^p(\Omega)) \cap C^{s-1}([0, T]; W_K^{2,p}(\Omega)). \end{cases} \quad (2.36)$$

Moreover, we list some further consistency conditions:

$$(C2,D) \quad v_0(x) = 0, \quad \forall x \in \partial\Omega, \quad (2.37)$$

$$(C2,N) \quad \frac{\partial v_0}{\partial \nu}(x) = 0, \quad \forall x \in \partial\Omega, \quad (2.38)$$

$$\Phi[v_0](r) = D_t g_1(0, r) - \Phi[D_t u_1(0, \cdot)](r), \quad \forall r \in (R_1, R_2), \quad (2.39)$$

$$\Psi[v_0] = D_t g_2(0) - \Psi[D_t u_1(0, \cdot)], \quad (2.40)$$

where

$$v_0(x) = \mathcal{A}u_0(x) + f(0, x) - D_t u_1(0, x), \quad \forall x \in \Omega. \quad (2.41)$$

**Theorem 2.6.** *Let the coefficients  $a_{i,j}$  ( $i, j = 1, 2, 3$ ) be represented by (1.3) where the functions  $a, b, c, d$  satisfy (1.4), (1.5). Moreover, let assumptions (2.1), (2.10) – (2.15) be fulfilled and assume that the data enjoy properties (2.17) – (2.23) and satisfy (2.30), (2.31) and the consistency conditions (C1, K) (cf. (1.16), (1.17)), (C2, K) as well as (1.18), (1.19), (2.39), (2.40).*

*Then there exists  $T^* \in (0, T]$  such that the identification problem  $P(K)$ ,  $K \in \{D, N\}$ , admits a unique solution  $(u, k) \in \mathcal{U}^{2,p}(T^*) \times C^\beta([0, T^*], W_2^{1,p}(0, R))$  depending continuously on the data with respect to the norms pointed out in (2.17) – (2.23).*

*In the case of the specific operators  $\Phi, \Psi$  defined by (1.14), (1.15) the previous results are still true if  $\psi \in C^1(\overline{\Omega})$ , with  $\psi|_{\partial\Omega} = 0$  when  $K = D$ .*

**Corollary 2.7.** *When  $\Phi$  and  $\Psi$  are defined by (1.14) and (1.15), respectively, and the coefficients  $a_{i,j}$  ( $i, j = 1, 2, 3$ ) are represented by (1.3), conditions (2.11) – (2.15) are satisfied under assumptions (1.4), (2.10) and the hypothesis  $\psi \in C^1(\overline{\Omega})$ , with  $\psi|_{\partial\Omega} = 0$  when  $K = D$ .*

*Proof.* From definitions (1.15) and Hölder's inequality it immediately follows

$$|\Psi[v]| \leq \|\psi\|_{C(\overline{\Omega})} \|v\|_{L^1(\Omega)} \leq \left[ \frac{4}{3} \pi R^3 \right]^{1/p'} \|\psi\|_{C(\overline{\Omega})} \|v\|_{L^p(\Omega)}. \quad (2.42)$$

Hence, from (2.6) and (2.42) we have that (2.11) is satisfied. Definition (1.14) easily implies (2.12) and (2.13), as we have already noted in (2.5). So, it remains only to prove that decompositions (2.14) and (2.15) hold.

When the coefficients  $a_{i,j}$  are represented by (1.3) the second-order differential operator  $\mathcal{A}$  can be represented, in spherical co-ordinates, by operator  $\tilde{\mathcal{A}}$  defined by (2.2). Our next task consists in computing  $\Phi[\tilde{\mathcal{A}}w]$  for any  $w \in W_K^{2,p}(\Omega)$ ,  $p \in (3, +\infty)$ . Observe first that from (2.6) and (2.16) it follows

$$\Phi[\tilde{\mathcal{A}}_1 w](r) = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} \lambda(Rx') \left\{ D_r[h(r)D_r w(rx')] + 2\frac{h(r)}{r} D_r w(rx') \right\} d\varphi = \tilde{\mathcal{A}}_1 \Phi[w](r) \quad (2.43)$$

Since  $p \in (3, +\infty)$ , using the Sobolev embedding theorem of  $W^{1,p}(\Omega)$  into  $C(\overline{\Omega})$  and the well-known formulae

$$\begin{cases} D_r = \cos\varphi \sin\theta D_{x_1} + \sin\varphi \sin\theta D_{x_2} + \cos\theta D_{x_3}, \\ D_\varphi = -r \sin\varphi \sin\theta D_{x_1} + r \cos\varphi \sin\theta D_{x_2}, \\ D_\theta = r \cos\varphi \cos\theta D_{x_1} + r \sin\varphi \sin\theta D_{x_2} - r \sin\theta D_{x_3}, \end{cases} \quad (2.44)$$

it can be easily shown that  $(D_\varphi w)/(r \sin\theta)$  and  $(D_\theta w)/r$  are bounded, while the functions  $(D_\varphi^2 w)/\sin\theta$  and  $D_\theta(\sin\theta D_\theta w)$  belong to  $L^1(\partial B(0, r))$  for every  $r \in (0, R)$ . Therefore,



integrating by parts, we obtain

$$\begin{aligned} \Phi \left[ \frac{a(r) + b(r)}{r^2 \sin \theta} \left( (\sin \theta)^{-1} D_\varphi^2 w + D_\theta (\sin \theta D_\theta w) \right) \right] (r) \\ = \frac{a(r) + b(r)}{r^2} \left\{ \int_0^\pi \left[ \frac{D_\varphi w(r x')}{\sin \theta} \right]_{\varphi=0}^{\varphi=2\pi} d\theta + \int_0^{2\pi} \left[ D_\theta w(r x') \sin \theta \right]_{\theta=0}^{\theta=\pi} d\varphi \right\} = 0, \end{aligned} \quad (2.45)$$

$$\begin{aligned} \Phi \left[ \frac{1}{r^2 \sin \theta} \left( (\sin \theta)^{-1} D_\varphi [\tilde{c}(r, \varphi, \theta) D_\varphi w] + D_\theta [\tilde{c}(r, \varphi, \theta) \sin \theta D_\theta w] \right) \right] (r) \\ = \frac{1}{r^2} \left\{ \int_0^\pi \left[ \frac{\tilde{c}(r, \varphi, \theta) D_\varphi w(r x')}{\sin \theta} \right]_{\varphi=0}^{\varphi=2\pi} d\theta + \int_0^{2\pi} \left[ \tilde{c}(r, \varphi, \theta) D_\theta w(r x') \sin \theta \right]_{\theta=0}^{\theta=\pi} d\varphi \right\} = 0. \end{aligned} \quad (2.46)$$

Hence, from (2.43), (2.45), (2.46) we find that (2.14) holds for every  $w \in W_K^{2,p}(\Omega)$  with  $p \in (3, +\infty)$ .

Let now  $\Psi$  be the functional defined in (1.15). Analogously to what we have done for  $\Phi$ , we apply  $\Psi$  to both sides in (2.2). Performing computations similar to those made above and using the assumption  $\psi|_{\partial\Omega} = 0$  when  $K = D$  which ensure that the surface integral vanishes, we obtain the equation

$$\Psi[\tilde{\mathcal{A}}w] = \Psi_1[w], \quad w \in W_K^{2,p}(\Omega),$$

where

$$\begin{aligned} \Psi_1[w] = & - \int_0^R r^2 h(r) dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} D_r w(r x') D_r \psi(r x') d\varphi \\ & - \int_0^R r^2 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} [a(r) + b(r) + \tilde{c}(r, \varphi, \theta)] \frac{D_\varphi w(r x')}{r \sin \theta} \frac{D_\varphi \psi(r x')}{r \sin \theta} d\varphi \\ & - \int_0^R r^2 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} [a(r) + b(r) + \tilde{c}(r, \varphi, \theta)] \frac{D_\theta w(r x')}{r} \frac{D_\theta \psi(r x')}{r} d\varphi. \end{aligned} \quad (2.47)$$

Now it is an easy task to show that  $\Psi_1$  defined in (2.47) belongs to  $W^{1,p}(\Omega)^*$ . Indeed, using formulae (2.44) and Hölder's inequality, we can easily find

$$|\Psi_1[w]| \leq C_1 \|\nabla u\|_{L^p(\Omega)} \leq C_1 \|w\|_{W^{1,p}(\Omega)}, \quad (2.48)$$

where  $C_1 > 0$  depends on  $\|\psi\|_{C^1(\overline{\Omega})}$  and  $\max[\|h\|_{L^\infty(0,R)}, \|a + b + c\|_{L^\infty(\Omega)}]$ , only.

Hence also decomposition (2.15) holds and this completes the proof.  $\square$

### 3 An equivalence result in the concrete case

Taking advantage of the results proved in [2], we limit ourselves to sketching the procedure for solving the necessary equivalence result.

We introduce the new triplet of unknown functions  $(v, l, q)$  defined by

$$v(t, x) = D_t u(t, x) - D_t u_1(t, x), \quad l(t) = k(t, R_2), \quad q(t, r) = D_r k(t, r), \quad (3.1)$$

so that  $u$  and  $k$  are given, respectively, by the following formulae

$$u(t, x) = u_1(t, x) - u_1(0, x) + u_0(x) + \int_0^t v(s, x) ds, \quad \forall (t, x) \in [0, T] \times \Omega, \quad (3.2)$$

$$k(t, r) = l(t) - \int_r^R q(t, \xi) d\xi := l(t) - Eq(t, r), \quad \forall (t, r) \in [0, T] \times (0, R). \quad (3.3)$$

Then problem (1.1), (1.9) – (1.13) can be shown to be equivalent to the following identification problem:

$$\begin{aligned} D_t v(t, x) &= \mathcal{A}v(t, x) + \int_0^t k(t-s, |x|) [\mathcal{B}v(s, x) + \mathcal{B}D_t u_1(s, x)] ds + k(t, |x|) \mathcal{B}u_0(x) \\ &\quad + \int_0^t D_{|x|} k(t-s, |x|) [\mathcal{C}v(s, x) + \mathcal{C}D_t u_1(s, x)] ds + D_{|x|} k(t, |x|) \mathcal{C}u_0(x) \\ &\quad + \mathcal{A}D_t u_1(t, x) - D_t^2 u_1(t, x) + D_t f(t, x), \quad \forall (t, x) \in [0, T] \times \Omega, \end{aligned} \quad (3.4)$$

$$v(0, x) = \mathcal{A}u_0(x) + f(0, \cdot) - D_t u_1(0, x) := v_0(x), \quad \forall x \in \Omega, \quad (3.5)$$

$$v \text{ satisfies the homogeneous boundary condition (K), } K \in \{D, N\}, \quad (3.6)$$

$$l(t) = l_0(t) + N_3(v, l, q)(t), \quad \forall t \in [0, T], \quad (3.7)$$

$$q(t, r) = q_0(t, r) + J_2(u_0)(r) N_3(v, l, q)(t) + N_2(v, l, q)(t, r), \quad \forall (t, r) \in [0, T] \times (0, R), \quad (3.8)$$

where we have set

$$l_0(t) := [J_1(u_0)]^{-1} N_0(u_0, u_1, g_1, g_2, f)(t), \quad \forall t \in [0, T], \quad (3.9)$$

$$q_0(t, r) := J_2(u_0)(r) h_0(t) + N_3^0(u_0, u_1, g_1, f)(t, r), \quad \forall (t, r) \in [0, T] \times (0, R). \quad (3.10)$$

We recall that operators  $J_0$ ,  $J_1$  and  $J_2$  are defined, respectively, by (2.30), (2.31) and

$$J_2(u_0)(r) = -\frac{\Phi[\mathcal{B}u_0](r)}{\Phi[\mathcal{C}u_0](r)} \exp \left[ \int_r^{R_2} \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right], \quad \forall r \in (0, R). \quad (3.11)$$

To define operators  $N_2$  and  $N_3$  appearing in (3.7), (3.8) we need to introduce the operators  $N_1$  and  $L$ :

$$\begin{aligned} N_1(v, l, q)(t, |x|) &:= -\int_0^t [l(t-s) - Eq(t-s, |x|)] [\mathcal{B}v(s, x) + \mathcal{B}D_t u_1(s, x)] ds \\ &\quad - \int_0^t q(t-s, |x|) [\mathcal{C}v(s, x) + \mathcal{C}D_t u_1(s, x)] ds, \quad \forall (t, x) \in [0, T] \times \Omega, \end{aligned} \quad (3.12)$$

$$Lg(t, r) := \int_r^{R_2} \exp \left[ \int_r^\eta \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right] \frac{g(t, \eta)}{\Phi[\mathcal{C}u_0](\eta)} d\eta, \quad \forall g \in L^1((0, T) \times (0, R)). \quad (3.13)$$

Now, denoting by  $I$  the identity operator, define  $N_2$  and  $N_3$  via the formulae

$$\begin{aligned} N_2(v, l, q)(t, r) &:= \frac{1}{\Phi[\mathcal{C}u_0](r)} [I + \Phi[\mathcal{B}u_0](r)L] \Phi[N_1(v, l, q)(t, \cdot)](r) \\ &:= J_3(u_0)(r) \Phi[N_1(v, l, q)(t, \cdot)](r), \end{aligned} \quad (3.14)$$

$$\begin{aligned} N_3(v, l, q)(t) &:= [J_1(u_0)]^{-1} \left\{ \Psi[N_1(v, l, q)(t, \cdot)] - \Psi[N_2(v, l, q)(t, \cdot)\mathcal{C}u_0] \right. \\ &\quad \left. + \Psi[E(N_2(v, l, q)(t, \cdot))\mathcal{B}u_0] - \Psi_1[v(t, \cdot)] \right\}, \end{aligned} \quad (3.15)$$

where  $\Psi_1$  is defined by (2.47).

Finally, to define operators  $N_0$  and  $N_3^0$  appearing in (3.9), (3.10) we need to introduce first the operators  $N_1^0$  and  $N_2^0$ , where operators  $\tilde{\mathcal{A}}$  and  $\tilde{\mathcal{A}}_1$  are defined, respectively, by (2.2) and (2.16):

$$\begin{aligned} N_1^0(u_1, g_1, f)(t, r) &= D_t^2 g_1(t, r) - D_t \tilde{\mathcal{A}}_1 g_1(t, r) \\ &\quad - \Phi[D_t f(t, \cdot)](r), \quad \forall (t, r) \in [0, T] \times (0, R), \end{aligned} \quad (3.16)$$

$$N_2^0(u_1, g_2, f)(t) = D_t^2 g_2(t) - \Psi_1[D_t u_1(t, \cdot)] - \Psi[D_t f(t, \cdot)], \quad \forall t \in [0, T]. \quad (3.17)$$

Then we define

$$\begin{aligned} N_3^0(u_0, u_1, g_1, f)(t, r) &:= \frac{1}{\Phi[\mathcal{C}u_0](r)} [I + \Phi[\mathcal{B}u_0](r)L] N_1^0(u_1, g_1, f)(t, r) \\ &:= J_3(u_0)(r) N_1^0(u_1, g_1, f)(t, r), \end{aligned} \quad (3.18)$$

$$\begin{aligned} N_0(u_0, u_1, g_1, g_2, f)(t) &:= N_2^0(u_1, g_2, f)(t) - \Psi[N_3^0(u_0, u_1, g_1, f)(t, \cdot)\mathcal{C}u_0] \\ &\quad - \Psi[E(N_3^0(u_0, u_1, g_1, f)(t, \cdot))\mathcal{B}u_0]. \end{aligned} \quad (3.19)$$

Finally, we introduce function  $k_0$  appearing in (2.21):

$$\begin{aligned} k_0(r) &= [J_1(u_0)]^{-1} \left\{ \Psi[\tilde{l}_2] + N_2^0(u_1, g_2, f)(0) - \Psi_1[v_0] \right\} \exp \left[ \int_r^{R_2} \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right] \\ &\quad + \int_{R_2}^r \exp \left[ \int_r^\eta \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right] \frac{N_1^0(u_1, g_1, f)(\eta)}{\Phi[\mathcal{C}u_0](\eta)} d\eta, \quad \forall r \in (R_1, R_2). \end{aligned} \quad (3.20)$$

where for any  $x \in \Omega$  we set

$$\begin{aligned} \tilde{l}_2(x) &:= \mathcal{C}u_0(x) \left\{ \frac{N_1^0(u_1, g_1, f)(|x|)}{\Phi[\mathcal{C}u_0](|x|)} - \frac{\Phi[\mathcal{B}u_0](|x|)}{\Phi[\mathcal{C}u_0](|x|)} \int_{R_2}^{|x|} \exp \left[ \int_{|x|}^\eta \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right] \right. \\ &\quad \left. \times \frac{N_1^0(u_1, g_1, f)(\eta)}{\Phi[\mathcal{C}u_0](\eta)} d\eta \right\} + \mathcal{B}u_0(x) \int_{R_2}^{|x|} \exp \left[ \int_{|x|}^\eta \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right] \frac{N_1^0(u_1, g_1, f)(\eta)}{\Phi[\mathcal{C}u_0](\eta)} d\eta. \end{aligned}$$

We can summarize the result sketched in this section in the following equivalence theorem.

**Theorem 3.1.** *The pair  $(u, k) \in \mathcal{U}^{2,p}(T) \times C^\beta([0, T]; W_2^{1,p}(0, R))$  is a solution to the identification problem P(K),  $K \in \{\mathbb{D}, \mathbb{N}\}$ , if and only if the triplet  $(v, l, q)$  defined by (3.1) belongs to  $\mathcal{U}_K^{1,p}(T) \times C^\beta([0, T]; \mathbb{R}) \times C^\beta([0, T]; L_2^p(0, R))$  and solves problem (3.4)–(3.8).*

## 4 An abstract formulation of problem (3.4)-(3.8).

Starting from the result of the previous section, we can reformulate our identification problem in a Banach space framework.

Let  $A : \mathcal{D}(A) \subset X \rightarrow X$  be a linear closed operator satisfying the following assumptions:

- (H1) *there exists  $\zeta \in (\pi/2, \pi)$  such that the resolvent set of  $A$  contains 0 and the open sector  $\Sigma_\zeta = \{\mu \in \mathbb{C} : |\arg \mu| < \zeta\}$ ;*
- (H2) *there exists  $M > 0$  such that  $\|(\mu I - A)^{-1}\|_{\mathcal{L}(X)} \leq M|\mu|^{-1}$  for every  $\mu \in \Sigma_\zeta$ ;*
- (H3)  *$X_1$  and  $X_2$  are Banach spaces such that  $\mathcal{D}(A) = X_2 \hookrightarrow X_1 \hookrightarrow X$ . Moreover,  $\mu \rightarrow (\mu I - A)^{-1}$  belongs to  $\mathcal{L}(X; X_1)$  and satisfies the estimate  $\|(\mu I - A)^{-1}\|_{\mathcal{L}(X; X_1)} \leq M|\mu|^{-1/2}$  for every  $\mu \in \Sigma_\zeta$ .*

Here  $\mathcal{L}(Z_1; Z_2)$  denotes, for any pair of Banach spaces  $Z_1$  and  $Z_2$ , the Banach space of all bounded linear operators from  $Z_1$  into  $Z_2$  equipped with the uniform-norm. In particular we set  $\mathcal{L}(X) = \mathcal{L}(X; X)$ .

By virtue of assumptions (H1), (H2) we can define the analytic semigroup  $\{e^{tA}\}_{t \geq 0}$  of bounded linear operators in  $\mathcal{L}(X)$  generated by  $A$ . As is well-known, there exist positive constants  $\tilde{c}_k(\zeta)$  ( $k \in \mathbb{N}$ ) such that

$$\|A^k e^{tA}\|_{\mathcal{L}(X)} \leq \tilde{c}_k(\zeta) M t^{-k}, \quad \forall t \in \mathbb{R}_+, \forall k \in \mathbb{N}.$$

After endowing  $\mathcal{D}(A)$  with the graph-norm, we can define the following family of interpolation spaces  $\mathcal{D}_A(\beta, p)$ ,  $\beta \in (0, 1)$ ,  $p \in [1, +\infty]$ , which are intermediate between  $\mathcal{D}(A)$  and  $X$ :

$$\mathcal{D}_A(\beta, p) = \left\{ x \in X : |x|_{\mathcal{D}_A(\beta, p)} < +\infty \right\}, \quad \text{if } p \in [1, +\infty], \quad (4.1)$$

where

$$|x|_{\mathcal{D}_A(\beta, p)} = \begin{cases} \left( \int_0^{+\infty} t^{(1-\beta)p-1} \|A e^{tA} x\|_X^p dt \right)^{1/p}, & \text{if } p \in [1, +\infty), \\ \sup_{0 < t \leq 1} (t^{1-\beta} \|A e^{tA} x\|_X), & \text{if } p = \infty. \end{cases} \quad (4.2)$$

They are well defined by virtue of assumption (H1). Moreover, we set

$$\mathcal{D}_A(1 + \beta, p) = \{x \in \mathcal{D}(A) : Ax \in \mathcal{D}_A(\beta, p)\}. \quad (4.3)$$

Consequently,  $\mathcal{D}_A(n + \beta, p)$ ,  $n \in \mathbb{N}$ ,  $\beta \in (0, 1)$ ,  $p \in [1, +\infty]$ , turns out to be a Banach space when equipped with the norm

$$\|x\|_{\mathcal{D}_A(n+\beta, p)} = \sum_{j=0}^n \|A^j x\|_X + |A^n x|_{\mathcal{D}_A(\beta, p)}. \quad (4.4)$$

In order to reformulate in an abstract form our identification problem(3.4)–(3.8) we need the following assumptions involving spaces, operators and data:

- (H4)  $Y$  and  $Y_1$  are Banach spaces such that  $Y_1 \hookrightarrow Y$ ;
- (H5)  $B : \mathcal{D}(B) \subset X \rightarrow X$  is a linear closed operator such that  $X_2 \subset \mathcal{D}(B)$ ;
- (H6)  $C : \mathcal{D}(C) := X_1 \subset X \rightarrow X$  is a linear closed operator;
- (H7)  $E \in \mathcal{L}(Y; Y_1)$ ,  $\Phi \in \mathcal{L}(X; Y)$ ,  $\Psi \in X^*$ ,  $\Psi_1 \in X_1^*$ ;
- (H8)  $\mathcal{M}$  is a continuous bilinear operator from  $Y \times \tilde{X}_1$  to  $X$  and from  $Y_1 \times X$  to  $X$ , where  $X_1 \hookrightarrow \tilde{X}_1$ ;
- (H9)  $J_1 : X_2 \rightarrow \mathbb{R}$ ,  $J_2 : X_2 \rightarrow Y$ ,  $J_3 : X_2 \rightarrow \mathcal{L}(Y)$  are three prescribed (non-linear) operators;
- (H10)  $u_0, v_0 \in X_2$ ,  $Cu_0 \in X_1$ ,  $J_1(u_0) \neq 0$ ,  $Bu_0 \in \mathcal{D}_A(\delta, +\infty)$ ,  $\delta \in (\beta, 1/2)$ ;
- (H11)  $q_0 \in C^\beta([0, T]; Y)$ ,  $l_0 \in C^\beta([0, T]; \mathbb{R})$ ;
- (H12)  $z_0 \in C^\beta([0, T]; X)$ ,  $z_1 \in C^\beta([0, T]; \tilde{X}_1)$ ,  $z_2 \in C^\beta([0, T]; X)$ ;
- (H13)  $Av_0 + \mathcal{M}(\tilde{q}_0, Cu_0) + \tilde{l}_0 Bu_0 - \mathcal{M}(E\tilde{q}_0, Bu_0) + z_2(0, \cdot) \in \mathcal{D}_A(\beta, +\infty)$ .

The elements  $\tilde{q}_0$  and  $\tilde{l}_0$  appearing in (H13) are defined by:

$$\begin{cases} \tilde{l}_0 = l_0(0) - [J_1(u_0)]^{-1} \Psi_1[v_0], \\ \tilde{q}_0 = q_0(0) + J_2(u_0) [J_1(u_0)]^{-1} \Psi_1[v_0], \end{cases} \quad (4.5)$$

where  $l_0$  and  $q_0$  are the elements appearing in (H11).

**Remark 4.1.** In the explicit case we get the equations

$$\tilde{l}_0 = k_0(R_2), \quad \tilde{q}_0(r) = k'_0(r). \quad (4.6)$$

where  $k_0$  is defined in (3.20).

We can now reformulate our direct problem: *determine a function  $v \in C^1([0, T]; X) \cap C([0, T]; X_2)$  such that*

$$\begin{aligned} v'(t) = & [\lambda_0 I + A]v(t) + \int_0^t l(t-s)[Bv(s) + z_0(s)]ds - \int_0^t \mathcal{M}(Eq(t-s), Bv(s) + z_0(s))ds \\ & + \int_0^t \mathcal{M}(q(t-s), Cv(s) + z_1(s))ds + \mathcal{M}(q(t), Cu_0) + l(t)Bu_0 \\ & - \mathcal{M}(Eq(t), Bu_0) + z_2(t), \quad \forall t \in [0, T], \end{aligned} \quad (4.7)$$

$$v(0) = v_0. \quad (4.8)$$

**Remark 4.2.** In the explicit case (3.4) – (3.8) we have  $A = \mathcal{A} - \lambda_0 I$ , with a large enough positive  $\lambda_0$ , and the functions  $z_0, z_1, z_2$  defined by

$$z_0 = D_t \mathcal{B}u_1, \quad z_1 = D_t \mathcal{C}u_1, \quad z_2 = D_t \mathcal{A}u_1 - D_t^2 u_1 + D_t f, \quad (4.9)$$

whereas  $v_0, h_0, q_0$  are defined, respectively, via the formulae (2.41), (3.9), (3.10).

Introducing the operators

$$\begin{aligned} \tilde{R}_2(v, h, q) := & -[J_1(u_0)]^{-1} \left\{ \Psi[\mathcal{M}(J_3(u_0)\Phi[N_1(v, l, q)], Cu_0)] \right. \\ & \left. - \Psi[\mathcal{M}(E(J_3(u_0)\Phi[N_1(v, l, q)], Bu_0)] - \Psi[N_1(v, l, q)] \right\}, \end{aligned} \quad (4.10)$$

$$\tilde{R}_3(v, h, q) := J_2(u_0)\tilde{R}_2(v, l, q) + J_3(u_0)\Phi[N_1(v, l, q)], \quad (4.11)$$

$$\begin{aligned} \tilde{S}_2(v) := & [J_1(u_0)]^{-1} \left\{ \Psi[\mathcal{M}(J_3(u_0)\Phi_1[v], Cu_0)] + \Psi[\mathcal{M}(E(J_3(u_0)\Phi_1[v], Cu_0)] - \Psi_1[v] \right\}, \\ & (4.12) \end{aligned}$$

$$\tilde{S}_3(v) := J_2(u_0)\tilde{S}_2(v), \quad (4.13)$$

the fixed-point system (3.7), (3.8) for  $l$  and  $q$  becomes

$$l = l_0 + \tilde{R}_2(v, l, q) + \tilde{S}_2(v), \quad (4.14)$$

$$q = q_0 + \tilde{R}_3(v, l, q) + \tilde{S}_3(v). \quad (4.15)$$

The present situation is analogous to the one in [3] (cf. Section 4). Consequently, also in this case we can apply the abstract results proved in [2] (cf. Sections 5 and 6) to get the following local in time existence and uniqueness theorem.

**Theorem 4.3.** *Under assumptions (H1) – (H13) there exists  $T^* \in (0, T)$  such that for any  $\tau \in (0, T^*]$  problem (4.7), (4.8), (4.14), (4.15) admits a unique solution  $(v, l, q) \in [C^{1+\beta}([0, \tau]; X) \cap C^\beta([0, \tau]; X_2)] \times C^\beta([0, \tau]; \mathbb{R}) \times C^\beta([0, \tau]; Y)$ .*

## 5 Solving the identification problem (3.4)–(3.8) and proving Theorem 2.6

The main difficulties we meet when we try to solve our identification problem  $P(K)$ ,  $K \in \{D, N\}$ , in the open ball  $\Omega$  can be overcome by introducing the representation (1.3) and the additional assumptions (1.4)–(1.5) for the coefficients  $a_{i,j}$  ( $i, j = 1, 2, 3$ ) of  $\mathcal{A}$ .

The basic result of this section is the following Theorem.

**Theorem 5.1.** *Let the coefficients  $a_{i,j}$  ( $i, j = 1, 2, 3$ ) be represented by (1.3) where the functions  $a, b, c, d$  satisfy (1.4)–(1.5). Moreover, let assumptions (2.1), (2.10)–(2.23), (2.30), (2.31) be fulfilled along with the consistency conditions (2.37)–(2.40).*

*Then there exists  $T^* \in (0, T]$  such that the identification problem (3.4) – (3.8) admits a unique solution  $(v, l, q) \in \mathcal{U}_K^{1,p}(T^*) \times C^\beta([0, T^*]; \mathbb{R}) \times C^\beta([0, T^*]; L_2^p(0, R))$  depending continuously on the data with respect to the norms pointed out in (2.17)–(2.23).*

*In the case of the specific operators  $\Phi, \Psi$  defined by (1.14), (1.15) the previous results are still true if  $\psi \in C^1(\overline{\Omega})$ , with  $\psi|_{\partial\Omega} = 0$  when  $K = D$ .*

*Proof.* We will show that under our assumption (1.3) – (1.5), (2.1) on the coefficients  $a_{i,j}$ ,  $b_{i,j}$ ,  $c_j$  ( $i, j = 1, 2, 3$ ) of the linear differential operators  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$  defined in (1.2) we can apply the abstract results of Section 4 to prove locally in time existence and uniqueness of the solution  $(u, k)$  to the identification problem  $P(K)$ ,  $K \in \{D, N\}$ .

For this purpose let  $p \in (3, +\infty)$  and let us choose the Banach space  $X$ ,  $\tilde{X}_1$ ,  $X_1$ ,  $X_2$ ,  $Y$ ,  $Y_1$  appearing in assumptions (H1) – (H12) according to the rule

$$X = L^p(\Omega), \quad \tilde{X}_1 = W^{1,p}(\Omega), \quad X_1 = W_K^{1,p}(\Omega), \quad X_2 = W_K^{2,p}(\Omega), \quad (5.1)$$

$$Y = L_2^p(0, R), \quad Y_1 = W_2^{1,p}(0, R). \quad (5.2)$$

Since  $p \in (3, +\infty)$ , reasoning as in the first part of Section 5 in [3], we conclude that  $A = \mathcal{A} - \lambda_0 I$  satisfies (H1) – (H3) in the sector  $\Sigma_\zeta$  for some  $\lambda_0 \in \mathbf{R}_+$ .

Since assumptions (H4) – (H6) are obviously fulfilled, we have that (H1) – (H6) hold. Define now operators  $\Phi$ ,  $\Psi$ ,  $\Psi_1$ , respectively, by (1.14), (1.15), (2.47) and operators  $E$  and  $\mathcal{M}$  by

$$Eq(r) = \int_r^{R_2} q(\xi) d\xi, \quad \forall r \in [0, R], \quad (5.3)$$

$$\mathcal{M}(q, w)(x) = q(|x|)w(x), \quad \forall x \in \Omega, \quad (5.4)$$

Then from Hölder's inequality and the fact that  $p \in (3, +\infty)$  we get

$$\begin{aligned} \|Eq\|_{L_2^p(0,R)}^p &= \int_0^R r^2 \left| \int_r^R q(\xi) d\xi \right|^p dr \leq \int_0^R r^2 \left[ \int_0^R \xi^{-2/p} \xi^{2/p} |q(\xi)| d\xi \right]^p dr \\ &\leq \|q\|_{L_2^p(0,R)}^p \int_0^R r^2 \left[ \int_0^R \xi^{-2/(p-1)} d\xi \right]^{p-1} dr = \frac{R^p}{3} \left( \frac{p-1}{p-3} \right)^{p-1} \|q\|_{L_2^p(0,R)}^p. \end{aligned} \quad (5.5)$$

Since  $D_r Eq(r) = -q(r)$  from (5.5) it follows:

$$\begin{aligned} \|Eq\|_{W_2^{1,p}(0,R)} &= \left[ \|Eq\|_{L_2^p(0,R)}^p + \|D_r Eq\|_{L_2^p(0,R)}^p \right]^{1/p} \\ &\leq \left[ \frac{R^p}{3} \left( \frac{p-1}{p-3} \right)^{p-1} + 1 \right]^{1/p} \|q\|_{L_2^p(0,R)}. \end{aligned} \quad (5.6)$$

Hence  $E \in \mathcal{L}(L_2^p(0, R); W_2^{1,p}(0, R))$ . Therefore, by virtue of (2.6), (2.42), (2.48) assumption (H7) is satisfied.

Since  $p \in (3, +\infty)$  we have the embedding (2.27). Then from the following inequalities,

$$\begin{aligned} \|\mathcal{M}(q, w)\|_{L^p(\Omega)}^p &= \int_\Omega |q(|x|)|^p |w(x)|^p dx \leq \|w\|_{C(\bar{\Omega})}^p \int_\Omega |q(|x|)|^p dx \\ &\leq 4\pi \|w\|_{C(\bar{\Omega})}^p \int_0^R r^2 |q(r)|^p dx \leq C \|w\|_{W^{1,p}(\Omega)}^p \|q\|_{L_2^p(0,R)}^p, \end{aligned} \quad (5.7)$$

we conclude that  $\mathcal{M}$  is a bilinear continuous operator from  $L_2^p(0, R) \times W^{1,p}(\Omega)$  to  $L^p(\Omega)$ . Moreover, using the embedding (2.28) it is an easy task to prove that  $\mathcal{M}$  is also continuous

from  $W_2^{1,p}(0, R) \times L^p(\Omega)$  to  $L^p(\Omega)$  and so (H8) is satisfied.

Then we define  $J_1(u_0), J_2(u_0), J_3(u_0)$  according to formulae (2.31), (3.11), (3.14) and it immediately follows that assumptions (H9) is satisfied, too.

Finally we estimate the vector  $(v_0, z_0, z_1, z_2, h_0, q_0)$  in terms of the data  $(f, u_0, u_1, g_1, g_2)$ . Definitions (3.16)–(3.19) imply that

$$\begin{aligned} N_1^0(u_1, g_1, f), N_3^0(u_0, u_1, g_1, f) &\in C^\beta([0, T]; L_2^p(0, R)), \\ N_2^0(u_1, g_2, f), N_0(u_0, u_1, g_1, g_2, f) &\in C^\beta([0, T]). \end{aligned}$$

Therefore from (3.9) and (3.10) we deduce

$$(h_0, q_0) \in C^\beta([0, T]) \times C^\beta([0, T]; L_2^p(0, R)), \quad (5.8)$$

whereas from (4.9), (3.5) and hypotheses (2.17)–(2.21) it follows

$$(z_0, z_1, z_2) \in C^\beta([0, T]; L^p(\Omega)) \times C^\beta([0, T]; W^{1,p}(\Omega)) \times C^\beta([0, T]; L^p(\Omega)), \quad (5.9)$$

$$v_0 \in W_K^{2,p}(\Omega), \quad \mathcal{A}v_0 + z_2(0, \cdot) \in W_K^{2\beta,p}(\Omega). \quad (5.10)$$

Hence assumptions (H10)–(H12) are also satisfied. To check condition (H13) first we recall that in this case the interpolation space  $\mathcal{D}_A(\beta, +\infty)$  coincides with the Besov spaces  $B_{H,K}^{2\beta,p,\infty}(\Omega) \equiv (L^p(\Omega), W_{H,K}^{2,p}(\Omega))_{\beta,\infty}$  (cf. [5, section 4.3.3]). Moreover, we recall that  $B_{H,K}^{2\beta,p,p}(\Omega) = W_{H,K}^{2\beta,p}(\Omega)$ . Finally, we remind the basic inclusion (cf. [5, section 4.6.1])

$$W^{s,p}(\Omega) \hookrightarrow B^{s,p,\infty}(\Omega), \quad \text{if } s \notin \mathbb{N}. \quad (5.11)$$

Since our function  $F$  defined in (2.21) belongs to  $W_{H,K}^{2\beta,p}(\Omega)$ , it is necessarily an element of  $B_{H,K}^{2\beta,p,\infty}(\Omega)$ . Therefore (H13) is satisfied, too.  $\square$

**Proof of Theorem 2.6.** It easily follows from Theorems 3.1 and 5.1.  $\square$

**Remark 5.2.** We want here to give some insight into the somewhat involved condition (2.21). For this purpose we need to assume that the functions  $a, b, d \in W^{3,\infty}((0, R))$ ,  $c \in W^{3,\infty}(\Omega)$  satisfy the following conditions

$$b(0) = b'(0) = b''(0) = 0, \quad d(0) = d'(0) = d''(0) = 0,$$

$$a'(0) = a''(0) = 0, \quad D_{x_i}c(0) = D_{x_i}D_{x_j}c(0) = 0, \quad i, j = 1, \dots, n.$$

This implies that the coefficients  $a_{i,j}$  belongs  $W^{3,\infty}(\Omega)$ ,  $i, j = 1, 2, 3$ . Then we observe that function  $k_0$  defined in (3.20) actually belongs to  $C^{1+\alpha}([R_1, R_2])$ ,  $\alpha \in (2\beta, 1)$ . It is then an easy task to show the membership of function  $F$  in  $W_{H,K}^{2\beta,p}(\Omega)$ ,  $\beta \in (0, 1/2)$  under the following regularity assumptions

- i) for any  $\rho \in C^\alpha(\overline{\Omega})$ ,  $\alpha \in (2\beta, 1)$ ,  $w \in W^{2\beta,p}(\Omega)$ ,  $\rho w \in W^{2\beta,p}(\Omega)$  and satisfies the estimate  $\|\rho w\|_{W^{2\beta,p}(\Omega)} \leq C\|\rho\|_{C^\alpha(\overline{\Omega})}\|w\|_{W^{2\beta,p}(\Omega)}$ ;
- ii) operator  $\Phi$  maps  $C^\alpha(\overline{\Omega})$  into  $C^\alpha([R_1, R_2])$ .



As for as the boundary conditions involved by assumption (H13) are concerned, we observe that they are missing when  $(K) = (N)$ , while in the remaining case they are so complicated that we like better not to explicit them and we limit to list them as

$F$  satisfies boundary conditions  $(K)$ .

Of course, when needed, such conditions can be explicitly computed in terms of the data and function  $k_0$  defined in (3.20).

## 6 The two-dimensional case

In this section we deal with the planar identification problem  $P(K)$  related to the disk  $\Omega = \{x \in \mathbb{R}^2 : |x| < R\}$  where  $R > 0$ .

Operators  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$  are defined by (1.2) simply replacing the subscript 3 with 2:

$$\mathcal{A} = \sum_{j=1}^2 D_{x_j} \left( \sum_{k=1}^2 a_{j,k}(x) D_{x_k} \right), \quad \mathcal{B} = \sum_{j=1}^2 D_{x_j} \left( \sum_{k=1}^2 b_{j,k}(x) D_{x_k} \right), \quad \mathcal{C} = \sum_{j=1}^2 c_j(x) D_{x_j}. \quad (6.1)$$

According to (1.3) for the two-dimensional case, we assume that the coefficients  $a_{i,j}$  of  $\mathcal{A}$  have the following representation

$$\begin{cases} a_{1,1}(x) = a(|x|) + \frac{x_2^2[c(x) + b(|x|)]}{|x|^2} - \frac{x_1^2 d(|x|)}{|x|^2}, \\ a_{2,2}(x) = a(|x|) + \frac{x_1^2[c(x) + b(|x|)]}{|x|^2} - \frac{x_2^2 d(|x|)}{|x|^2}, \\ a_{1,2}(x) = a_{2,1}(x) = -\frac{x_1 x_2 [b(|x|) + c(x) + d(|x|)]}{|x|^2}, \end{cases} \quad (6.2)$$

where the function  $a$ ,  $b$ ,  $c$  and  $d$  satisfy properties (1.4), (1.5).

Furthermore we assume that the coefficients of operators  $\mathcal{B}$ ,  $\mathcal{C}$  satisfy (2.1).

In the two-dimensional case, setting  $x' = (\cos\varphi, \sin\varphi)$  an example of admissible linear operators  $\Phi$  and  $\Psi$  is now the following:

$$\Phi[v](r) := \int_0^{2\pi} v(rx') d\varphi, \quad (6.3)$$

$$\Psi[v] := \int_0^R r dr \int_0^{2\pi} \psi(rx') v(rx') d\varphi, \quad (6.4)$$

Similarly to (2.2), using (6.2), we obtain the following polar representation for the second order differential operator  $\mathcal{A}$ :

$$\tilde{\mathcal{A}} = D_r [h(r) D_r] + \frac{h(r) D_r}{r} + \frac{a(r) + b(r)}{r^2} D_\varphi^2 + \frac{1}{r^2} D_\varphi [\tilde{c}(r, \varphi) D_\varphi], \quad (6.5)$$

where  $\tilde{c}(r, \varphi) = c(r \cos\varphi, r \sin\varphi)$  and function  $h$  is defined in (1.6).

Working in the Sobolev spaces  $W^{k,p}(\Omega)$ , we will assume

$$p \in (2, +\infty). \quad (6.6)$$

Moreover, our assumptions on operators  $\Phi$  and  $\Psi$  and the data will be the same as in (2.11)–(2.23) with the spaces  $L_2^p(0, R)$  and  $W_2^{2,p}(0, R)$  replaced, respectively, by  $L_1^p(0, R)$  and  $W_1^{2,p}(0, R)$ . The Banach spaces  $\mathcal{U}^{s,p}(T)$ ,  $\mathcal{U}_K^{s,p}(T)$  are still defined by (2.36).

**Theorem 6.1.** *Let us suppose that the coefficients  $a_{i,j}$  ( $i, j = 1, 2$ ) are represented by (6.2) and that (1.4), (1.5), (2.1), (2.11)–(2.15), (6.6) are fulfilled. Moreover, assume that the data enjoy the properties (2.17)–(2.23) and satisfy inequalities (2.30), (2.31) as well as consistency conditions (1.16) – (1.19), (2.37) – (2.40).*

*Then there exists  $T^* \in (0, T]$  such that the identification problem  $P(K)$ ,  $K \in \{D, N\}$ , admits a unique solution  $(u, k) \in \mathcal{U}^{2,p}(T^*) \times C^\beta([0, T^*]; W_1^{1,p}(0, R))$  depending continuously on the data with respect to the norms pointed out in (2.17)–(2.23).*

*In the case of the specific operators  $\Phi$ ,  $\Psi$  defined as in (6.3), (6.4) the previous results are still true if we assume  $\psi \in C^1(\overline{\Omega})$  with  $\psi|_{\partial\Omega} = 0$  when  $K = D$ .*

**Lemma 6.2.** *When  $\Phi$  and  $\Psi$  are defined by (6.3) and (6.4), respectively, and the coefficients  $a_{i,j}$  ( $i, j = 1, 2$ ) are represented by (6.2), conditions (2.11)–(2.15) are satisfied under assumptions (1.4), (6.6) and the hypothesis  $\psi \in C^1(\overline{\Omega})$  with  $\psi|_{\partial\Omega} = 0$  when  $K = D$ .*

*Proof.* It is essentially the same as that of Lemma 2.7. Therefore we leave it to the reader.  $\square$

For the two-dimensional case the results of Section 5 are still true. Therefore the proof of Theorem 5.1 is analogous to the one of Theorem 2.6.

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